7

DISCRETE CONVOLUTION AND CORRELATION

Possibly the most important discrete Fourier transform properties are those of convolution and correlation. This follows because the importance of the fast Fourier transform is primarily a result of its efficiency in computing discrete convolution or correlation. In this chapter, we examine, analytically and graphically, the discrete convolution and correlation equations. The relationship between discrete and continuous convolution is also explored in detail.

7.1 DISCRETE CONVOLUTION

Discrete convolution is defined by the summation:

$$y(kT) = \sum_{i=0}^{N-1} x(iT)h[(k - i)T] \tag{7.1}$$

where both $x(kT)$ and $h(kT)$ are periodic functions with period N,

$$\begin{aligned} x(kT) &= x[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \\ h(kT) &= h[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{7.2}$$

For convenience of notation, discrete convolution is normally written as

$$y(kT) = x(kT) * h(kT) \tag{7.3}$$

To examine the discrete convolution equation, consider the illustrations of Fig. 7.1. Both functions $x(kT)$ and $h(kT)$ are periodic with period

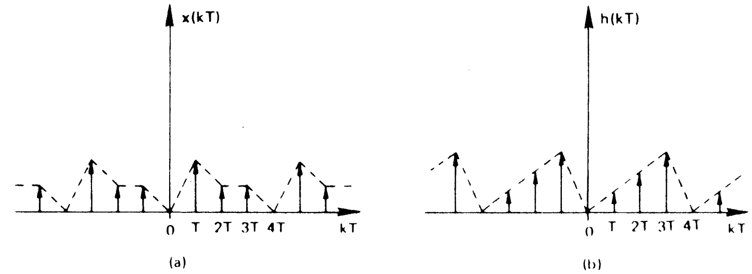

Figure 7.1 Example sampled waveforms to be convolved discretely.

$N = 4$. From Eq. (7.1), functions $x(iT)$ and $h[(k - i)T]$ are required. Function $h(-iT)$ is the image of $h(iT)$ about the ordinate axis, as illustrated in Fig. 7.2(a); function $h[(k - i)T]$ is simply the function $h(-iT)$ shifted by the amount kT. Figure 7.2(b) illustrates $h[(k - i)T]$ for the shift $2T$. Equation (7.1) is evaluated for each kT shift by performing the required multiplications and additions.

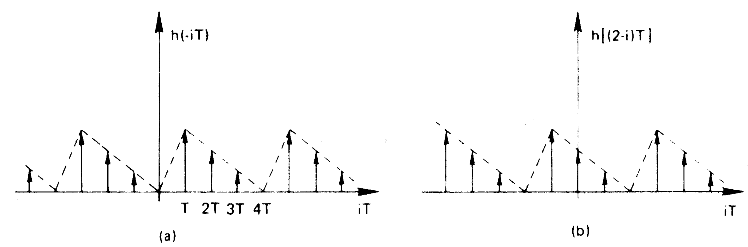

Figure 7.2 Graphical description of discrete convolution shifting operation.

7.2 GRAPHICAL INTERPRETATION OF DISCRETE CONVOLUTION

The discrete convolution process is illustrated graphically in Fig. 7.3. Sample values of $x(kT)$ and $h(kT)$ are denoted by *dots* and *crosses*, respectively. Figure 7.3(a) illustrates the desired computation for $k = 0$. The value of each *dot* is multiplied by the value of the *cross* that occurs at the same abscissa value; these products are summed over the $N = 4$ discrete values indicated. Computation of Eq. (7.1) is graphically evaluated for $k = 1$ in Fig. 7.3(b); multiplication and summation is over the N points indicated. Figures 7.3(c) and (d) illustrate the convolution computation for $k = 2$ and $k = 3$, respectively. Note that for $k = 4$ [Fig. 7.3(e)], the terms multiplied and summed are identical to those of Fig. 7.3(a). This is expected because

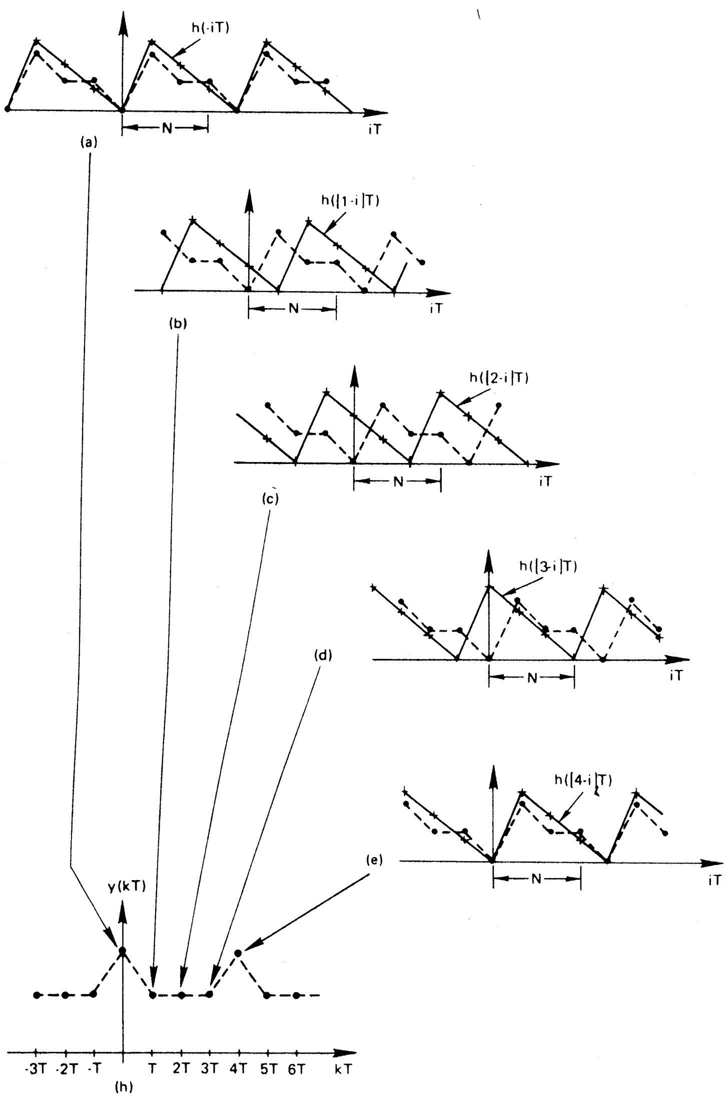

Figure 7.3 Graphical illustration of discrete convolution.

both $x(kT)$ and $h(kT)$ are periodic with a period of four terms. Therefore,

$$y(kT) = y[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \tag{7.4}$$

Steps for graphically computing the discrete convolution differ from those of continuous convolution only in that integration is replaced by summation. For discrete convolution, these steps are (1) folding, (2) displacement or shifting, (3) multiplication, and (4) summation. As in the convolution of continuous functions, either the sequence $x(kT)$ or $h(kT)$ can be selected for displacement. Equation (7.1) can be written equivalently as

$$y(kT) = \sum_{i=0}^{N-1} x[(k - i)T]h(iT) \tag{7.5}$$

7.3 RELATIONSHIP BETWEEN DISCRETE AND CONTINUOUS CONVOLUTION

If we only consider periodic functions represented by equally spaced impulse functions, discrete convolution relates identically to its continuous equivalent. This follows because, as we show in Appendix A (Eq. A.14), continuous convolution is well-defined for impulse functions.

The most important application of discrete convolution is not to sampled periodic functions but rather to approximate the continuous convolutions of general waveforms. For this reason, we will now explore in detail the relationship between discrete and continuous convolution.

Discrete Convolution of Finite-Duration Waveforms

Consider the functions $x(t)$ and $h(t)$, as illustrated in Fig. 7.4(a). We wish to convolve these two functions both continuously and discretely and to compare these results. Continuous convolution $y(t)$ of the two functions is also shown in Fig. 7.4(a). To evaluate the discrete convolution, we sample both $x(t)$ and $h(t)$ with sample interval T and we assume that both sample functions are periodic with period N. As illustrated in Fig. 7.4(b), the period has been chosen as $N = 9$ and both $x(kT)$ and $h(kT)$ are represented by $P = Q = 6$ samples; the remaining samples defining a period are set to zero. Figure 7.4(b) also illustrates the discrete convolution $y(kT)$ for the period $N = 9$; for this choice of N, the discrete convolution is a very poor approximation of the continuous case because the periodicity constraint results in an overlap of the desired periodic output. That is, we did not choose the period sufficiently large so that the convolution result of one period would not *interfere* or *overlap* the convolution result of the succeeding period. It is obvious that if we wish the discrete convolution to approximate continuous

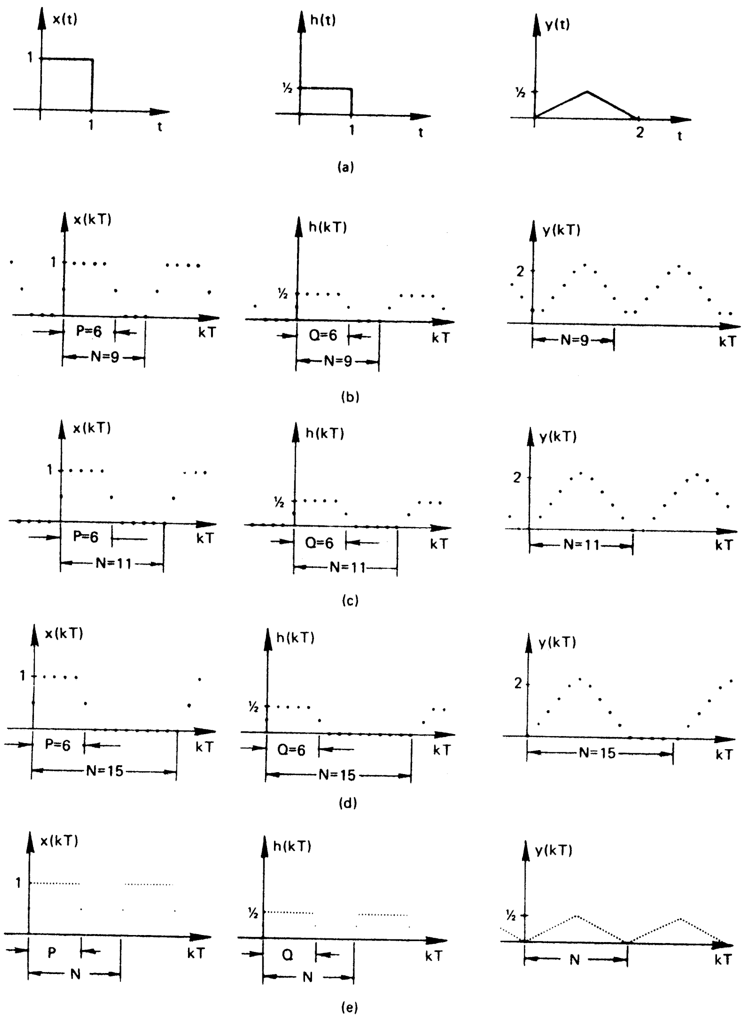

Figure 7.4 Relationship between discrete and continuous convolution: finite-duration waveforms.

convolution, then it is necessary that the period be chosen so that there is no overlap.

Choose the period according to the relationship

$$N = P + Q - 1 \tag{7.6}$$

This situation is illustrated in Fig. 7.4(c), where $N = P + Q - 1 = 11$.

Note that for this choice of N there is no overlap in the resulting convolution. Equation (7.6) is based on the fact that the convolution of a function represented by P samples and a function represented by Q samples is a function described by $P + Q - 1$ samples.

There is no advantage in choosing $N > P + Q - 1$; as shown in Fig. 7.4(d), for $N = 15$, the nonzero values of the discrete convolution are identical to those of Fig. 7.4(c). As long as N is chosen according to Eq. (7.6), discrete convolution results in a periodic function, where each period approximates the continuous convolution results.

Figure 7.4(c) illustrates the fact that discrete convolution results are scaled differently than that of continuous convolution. This scaling constant is T; modifying the discrete convolution Eq. (7.1), we obtain

$$y(kT) = T \sum_{i=0}^{N-1} x(iT)h[(k - i)T] \tag{7.7}$$

The relationship of Eq. (7.7) is simply the continuous convolution integral for time-limited functions evaluated by rectangular integration. Thus, for finite-length time functions, discrete convolution approximates continuous convolution within the error introduced by rectangular integration. As illustrated in Fig. 7.4(e), if the sample interval T is made sufficiently small, then the error introduced by the discrete convolution Eq. (7.7) is negligible.

Example 7.1 Circular Convolution

Discrete convolution yields periodic results because of the periodicity of the functions being convolved. This periodicity gives rise to what is commonly called *circular convolution*. Figure 7.5 illustrates this concept.

In Figure 7.5(a), we show two example discrete periodic waveforms to be convolved. For the shift $k = 2$, Fig. 7.5(b) illustrates the appropriate folding and shifting operations. Multiplication and addition over the $N = 8$ points of the period yield the convolution results for $k = 2$. An alternate way of displaying the discrete convolution of Fig. 7.5(b) for shift $k = 2$ is shown in Fig. 7.5(c). The rings represent one period of the two periodic functions; the inner ring is $h(iT)$ and is the function being shifted. As illustrated, the function is set for a shift of $k = 2$. The outer ring corresponds to the function $x(iT)$. Appropriate values to be multiplied are adjacent to each other. These multiplied results are then summed around the circle (i.e., over one period).

The inner ring is turned for each shift of k. As the ring is turned, it returns to its original position every eight shifts. Hence, the same values will be computed. This corresponds to the periodic convolution results discussed previously. Figure 7.5(c) can also be used to illustrate the problem of overlap. As the inner ring turns, there must be a sufficient number of zero values in the outer ring so that a convolution value is not computed, which is a function of both ends of the data used to form the outer ring. If sufficient zeros are appended to the nonzero sample values of each function to be convolved, then the finite-duration convolution result does not overlap with the following period.

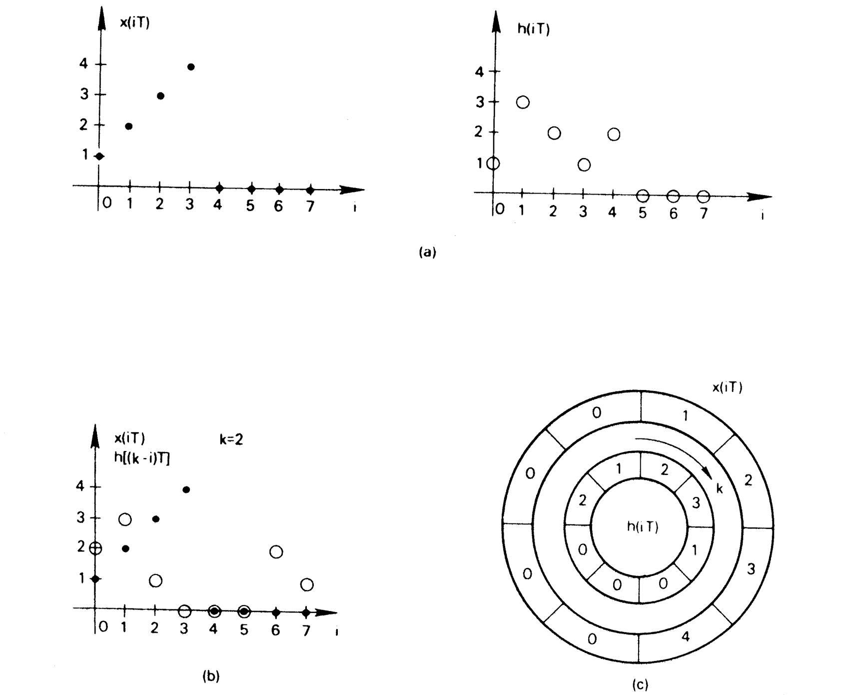

Figure 7.5 Graphical illustration of *circular convolution*.

Discrete Convolution of an Infinite- and a Finite-Duration Waveform

The previous example considered the case for which both $x(kT)$ and $h(kT)$ were of finite duration. Another case of interest is that where only one of the time functions to be convolved is finite. To explore the relationship of the discrete and continuous convolution for this case, consider the illustrations of Fig. 7.6. As illustrated in Fig. 7.6(a), function $h(t)$ is assumed to be of finite duration and $x(t)$ of infinite duration; convolution of these two functions is shown in Fig. 7.6(b). Because the discrete convolution requires that both the sampled functions $x(kT)$ and $h(kT)$ be periodic, we obtain the illustrations of Fig. 7.6(c); period N has been chosen [Figs. 7.6(a) and (c)]. For $x(kT)$ infinite in duration, the imposed periodicity introduces what is known as an *end effect*.

Compare the discrete convolution of Fig. 7.6(d) and the continuous convolution [Fig. 7.6(b)]. As illustrated, the two results agree reasonably well, with the exception of the first $Q - 1$ samples of the discrete convolution. To establish this fact more clearly, consider the illustrations of Fig. 7.7. We show only one period of $x(iT)$ and $h[(5 - i)T]$. To compute the discrete convolution, Eq. (7.1), for this shift, we multiply those samples of $x(iT)$ and $h[(5 - i)T]$ that occur at the same time [Fig. 7.7(a)] and add. The convolution result is a function of $x(iT)$ at both ends of the period. Such a condition obviously has no meaningful interpretation in terms of the desired continuous convolution. Similar results are obtained for each shift value until the Q points of $h(iT)$ are shifted by $Q - 1$, that is, the end effect exists until shift $k = Q - 1$.

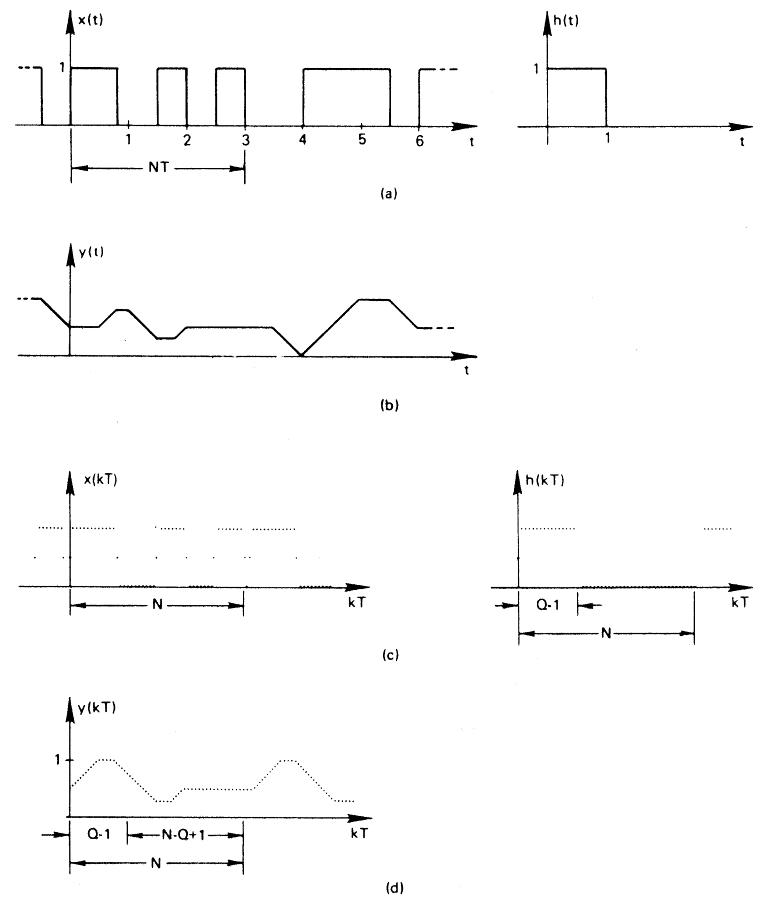

Figure 7.6 Relationship between discrete and continuous convolution: finite- and infinite-duration waveforms.

Note that the end effect does not occur at the right end of the N sample values; functions $h(iT)$ for the shift $k = N - 1$ (therefore maximum shift)

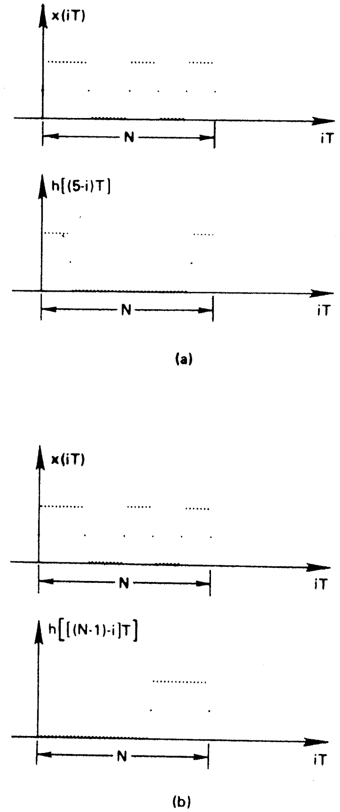

Figure 7.7 Illustration of the *end effect.*

and $x(iT)$ are illustrated in Fig. 7.7(b). Multiplication of those values of $x(iT)$ and $h[(N - 1 - i)T]$ that occur at the same time and subsequent addition yield the desired convolution; the result is only a function of the correct values of $x(iT)$.

If the sample interval T is chosen sufficiently small, then the discrete convolution for this example class of functions closely approximate the continuous convolution except for the end effect.

Summary

We have emphasized the point that discrete convolution is defined only for periodic functions. However, as illustrated graphically, the implications of this requirement are negligible if at least one of the functions to be convolved is of finite duration. For this case, discrete convolution is approximately equivalent to continuous convolution where the differences in the two methods are due to rectangular integration and to the end effect.

In general, it is impossible to discretely convolve two functions of infinite duration.

The convolution waveform illustrated could have been computed equivalently by means of the convolution theorem. Recall that the discrete convolution of Eq. (7.1) was defined in such a manner that those functions being convolved were assumed to be periodic. The underlying reason for this assumption is to enable the discrete convolution theorem, Eq. (6.50), to hold. If we compute the discrete Fourier transform of each of the periodic sequences $x(kT)$ and $h(kT)$, multiply the resulting transforms, and then compute the inverse discrete Fourier transform of this product, we obtain identical results to those illustrated. As is discussed in Chapter 10, it is normally faster computationally to use the discrete Fourier transform to compute the discrete convolution if the FFT is employed.

7.4 GRAPHICAL INTERPRETATION OF DISCRETE CORRELATION

Discrete correlation is defined as

$$z(kT) = \sum_{i=0}^{N-1} x(iT)h[(k + i)T] \tag{7.8}$$

where $x(kT)$, $h(kT)$, and $z(kT)$ are periodic functions.

$$\begin{aligned} z(kT) &= z[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \\ x(kT) &= x[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \\ h(kT) &= h[(k + rN)T] \qquad r = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{7.9}$$

As in the continuous case, discrete correlation differs from convolution in that there is no folding operation. Hence, the remaining rules for displacement, multiplication, and summation are performed exactly as for the case of discrete convolution.

To illustrate the process of discrete correlation or *lagged products*, as it sometimes is referred, consider Fig. 7.8. The discrete functions to be correlated are shown in Fig. 7.8(a). According to the rules for correlation, we shift, multiply, and sum, as illustrated in Figs. 7.8(b), (c), and (d), respectively. Compare with the results of Ex. 4.8. In Chapter 10, we discuss the application of the FFT for efficient computation of Eq. (7.8).

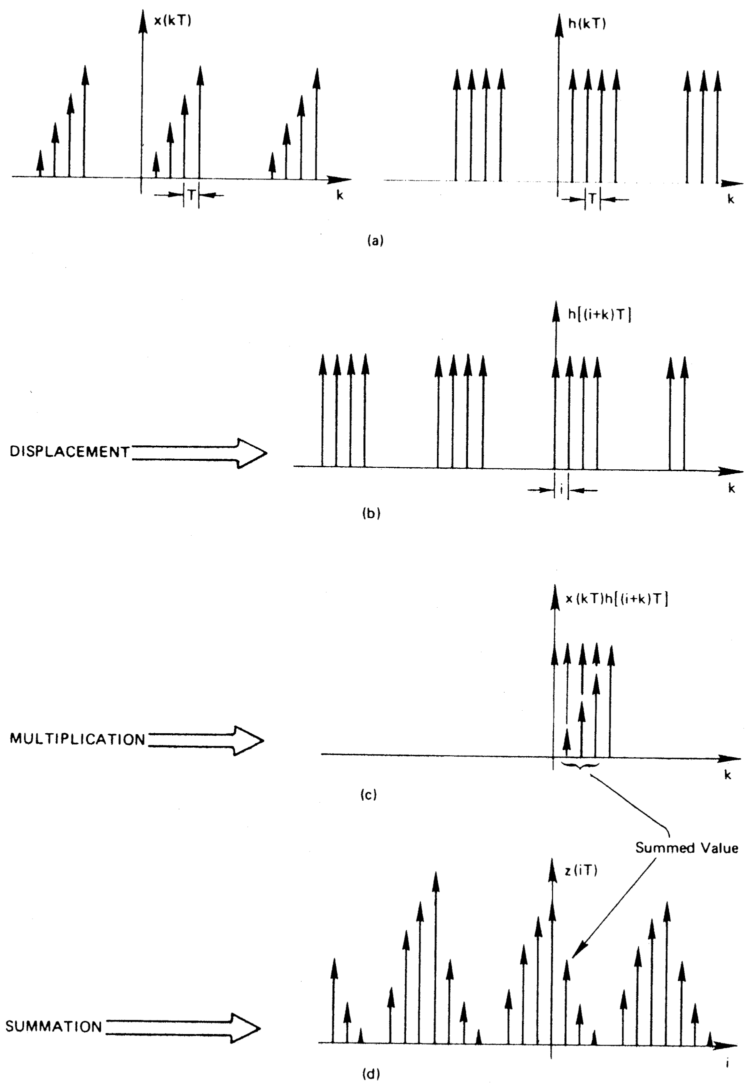

Figure 7.8 Graphical illustration of discrete correlation.

PROBLEMS

7.1. Let

$$\begin{aligned} x(kT) &= e^{-kT} & k &= 0, 1, 2, 3 \\ &= 0 & k &= 4, 5, \ldots, N \\ &= x[(k + rN)T] & r &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

and

$$\begin{aligned} h(kT) &= 1 & k &= 0, 1, 2 \\ &= 0 & k &= 3, 4, \ldots, N \\ &= h[(k + rN)T] & r &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

With $T = 1$, graphically and analytically determine $x(kT) * h(kT)$. Choose N less than, equal to, and greater than Eq. (7.6).

7.2. Consider the continuous functions $x(t)$ and $h(t)$, as illustrated in Fig. 4.14(a). Sample both functions with sample interval $T = T_0/4$ and assume both sample functions are periodic with period N. Choose N according to relationship of Eq. (7.6). Determine $x(kT) * h(kT)$ both analytically and graphically. Investigate the results of an incorrect choice of N. Compare results with continuous convolution results.

7.3. Repeat Problem 7.2 for Figs. 4.14(b) and (c).

7.4. Refer to Fig. 7.6. Let $x(t)$ be defined as illustrated in Fig. 7.6(a). Function $h(t)$ is given as

(a) $h(t) = \delta(t)$

(b) $h(t) = \delta(t) + \delta\left(t - \frac{3}{2}\right)$

(c)
$$\begin{aligned} h(t) &= 0 & t &< 0 \\ &= 1 & 0 < t &< \frac{1}{2} \\ &= 0 & \frac{1}{2} < t &< 1 \\ &= 1 & 1 < t &< \frac{3}{2} \\ &= 0 & t &> \frac{3}{2} \end{aligned}$$

Following Fig. 7.6, graphically determine the discrete convolution in each case. Compare the discrete and continuous convolution in each case. Investigate the end effect in each case.

7.5. It is desired to discretely convolve a finite-duration and an infinite-duration waveform. Assume that a hardware device is to be used that is limited in capacity to N sample values of each function. Describe a procedure that allows one to perform successive N-point discrete convolutions and combine the two to eliminate the end effect. Demonstrate your concept by repeating the illustrations of Fig. 7.6 for the case $NT = 1.5$. Successively apply the developed technique to determine the discrete convolution $y(kT)$ for $0 \le kT \le 3$.